Long-range random walk on percolation clusters

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Random walks on square-lattice percolation clusters are simulated for interaction ranges spanning one to *five* nearest-neighbor bonds ($R=1,\ldots,5$). The relative hopping probability is given by $\exp(-\alpha r)$, where r is the number of bonds traversed in one hop and α is a parameter ($0\leq\alpha\leq 10$). The fractal exponent for the random walks is universal. For $R=2$ (and $R=1$) we obtain a spectral dimension of $d_s=1.31\pm0.03$, in agreement with the Alexander-Orbach conjecture (1.333), and in even better agreement with the Aharony-Stauffer conjecture (1.309). Our results are based on the relation $d_s=(91/43)f$, where $S_n\sim N^f$ describes the mean number (S_N) of distinct sites visited in N steps for walks originating on *all* clusters. While the asymptotic limit of f is closely approached after 5000 nominal time steps for $\alpha=0$, much longer times (>50 000 steps) are required for $\alpha\gg0$. We also observe fractal-to-Euclidean crossovers above criticality; again, this crossover takes much longer for $\alpha\gg0$.

I. INTRODUCTION

Critical phenomena relating to long-range percolation[1–7] have been of much interest and some controversy. For the emergence of nonclassical critical exponents a sharp cutoff in interaction range seems to be of prime importance.[2,4] Such a cutoff also uniquely defines the cluster geometry. We report here a Monte Carlo simulation of random walks on such long-range percolation clusters of a square lattice with varying lengths of interaction range (see below) and with varying hopping probabilities within such an interaction range. It has been shown before[2,4] that the static critical exponents (β,γ,ν), and therefore the fractal dimension, obey the universality hypothesis, independent of interaction range (for finite-range cutoffs). The question posed here regards the universality of the dynamic exponents, e.g., the spectral dimension[8,9] (d_s), and the related question of fractal-to-Euclidean crossover.[10] This problem is also of much interest to experimental investigations of fractal-exciton trapping and reaction kinetics;[11–13] the triplet-exciton hopping in isotopic mixed crystals certainly involves such long-range random hops on percolation clusters, and it has been argued that the appropriate spectral dimension is the parameter of interest.

At the critical percolation concentration (p_c), the mean number (S_N) of distinct sites visited after N steps is believed[8–10] to obey the asymptotic expression

$$S_N\sim N^f \quad \text{as } N\to\infty \;, \tag{1}$$

where the relation of the fractal exponent f to the spectral dimension d_s is[14,15]

$$f=\tfrac{1}{2}d_s(1-\epsilon)=\tfrac{43}{91}d_s,\quad \epsilon=\beta/(\gamma+\beta)\;. \tag{2}$$

Here the critical exponents $\beta=\frac{5}{36}$ and $\gamma=\frac{43}{18}$ are assigned their usual two-dimensional values.[2–10] We note that Eq. (2) is for random walks that originate on *all* clusters (finite and infinite). We use the standard long-range cluster definition[2,4,11] which employs a "taxicab geometry,"[16] involving nearest-neighbor bonds in a site-percolation problem. The hopping probability P depends on the number r of such bonds required to connect two sites:

$$P\sim\exp(-\alpha r)\;, \tag{3}$$

where α is a parameter ranging upwards from $\alpha=0$ (constant probability). This functional dependence mimics the triplet exciton superexchange interactions and hopping probabilities[1–3,11] and is also close to some singlet exciton interactions.[11,17] The critical concentrations for the different interaction ranges ($R=1$–5) have been derived by Monte Carlo simulations[2,18] and from a position-space renormalization-group approach[4] and our best estimates for p_c are given in Table I.

We find that the fractal exponent f and the spectral dimension d_s have the same value, i.e., universal behavior, for all interaction ranges R. Our data agree better with the Aharony-Stauffer conjecture[8] ($d_s=1.309$) than with

TABLE I. Critical percolation concentration p_c as a function of the interaction range R (from Ref. 2).

R	p_c
1	0.5931
2	0.29
3	0.16
4	0.10
5	0.07
6	0.05

the Alexander-Orbach conjecture ($d_s = \frac{4}{3}$). Furthermore, we observe the expected[10] fractal-to-Euclidean crossovers above the critical percolation concentration. However, for large values of the parameter α, i.e., jump probabilities falling off sharply with distance r, the asymptotic behavior of Eq. (1) and the above "crossover" are reached only after much longer times than necessary for the ordinary case ($\alpha=0$) or for simple (short-range) percolation.[14,15]

II. METHOD OF SIMULATION

The same techniques used previously in random-walk simulations[14,15] are also employed here. The major difference is that now non-nearest-neighbor jumps are also possible. This point is treated as follows: Before every jump is to take place a search is made of all lattice sites enclosed in a radius of a given distance, where this distance is defined as the maximum range of interaction. In this search all open sites are identified together with their taxicab[16] distance from the point of origin. Then we apply the interaction potential as the case may be. For each potential, separately a jump probability matrix is constructed for every step that takes place. All steps are then performed at random, following their respective probability matrix elements. We monitor, as was done previously, the number, S_N, of distinct sites visited in an N-step walk, the mean-square displacement $\langle R_N^2 \rangle$, the probability of return to the origin P_0, etc. All simulations were performed on a 500×500 square lattice and were run on *all* clusters (using the long-range definition) and averaged over 1000 realizations, i.e., for each concentration (p) and potential (α), 1000 walks were performed over 5000 nominal time steps. For some selected cases the runs were much longer (Table II).

III. RESULTS

Figure 1 shows in a log-log representation the dependence of the mean number of distinct sites visited (S_n) on the number of steps (N) for the $R=2$ case ($\alpha=0$, equal probability for range-1 and range-2 hops) with walkers originating on all clusters. For the critical concentration

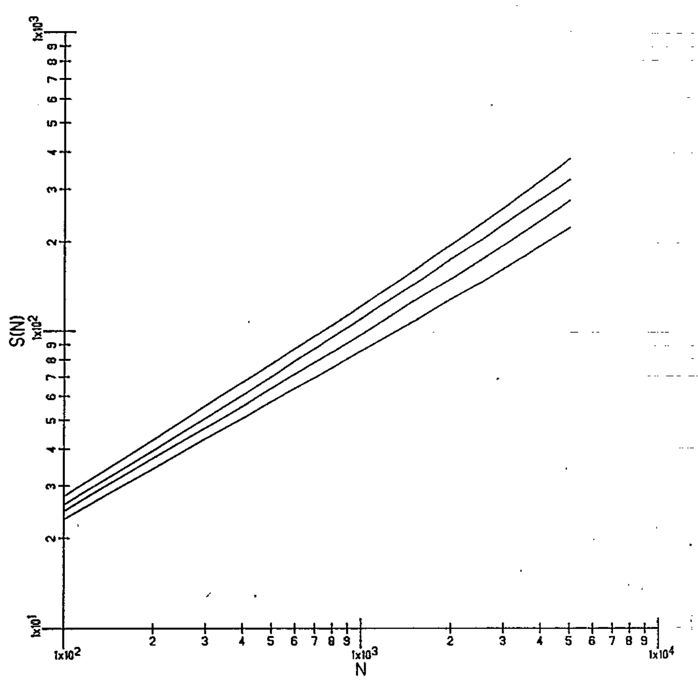

FIG. 1. Plot of $\ln S_N$ vs $\ln N$ for the case of $\alpha=0$, interaction range $R=2$ and four different concentrations $p, p=0.32$, 0.31, 0.30, and 0.29 (top to bottom). These are averages of 1000 realizations on 500×500 binary lattices at the above nominal concentrations. The random walks originate on any size cluster.

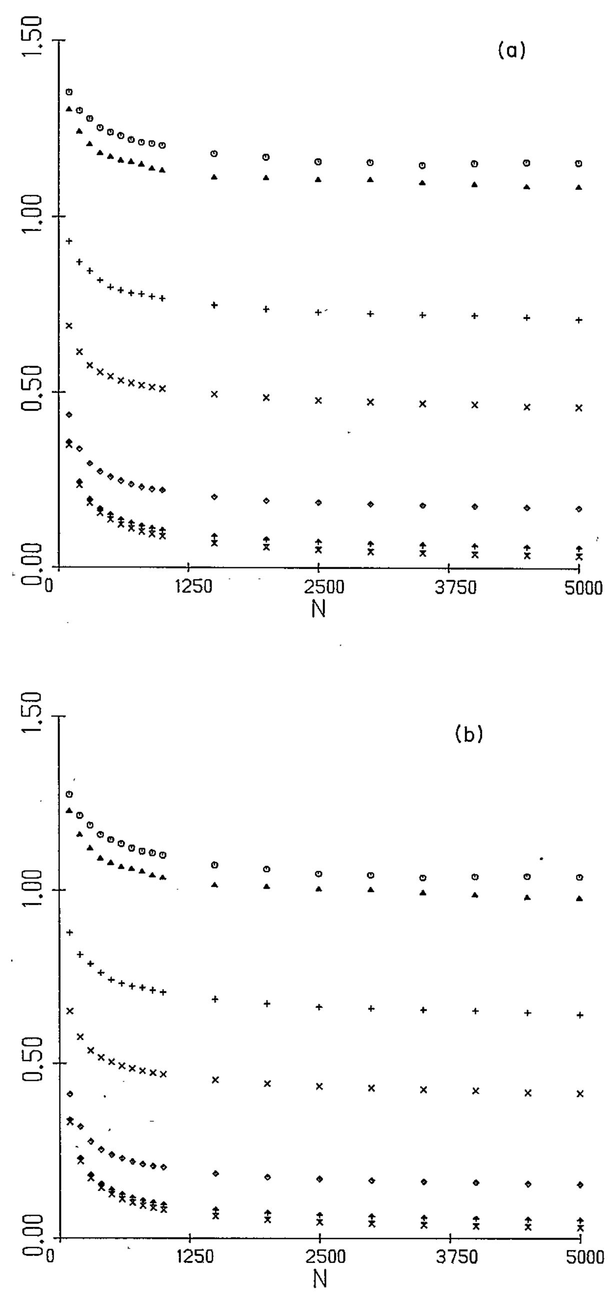

FIG. 2. (a) Plot of S_N/N^f vs N for different values of the parameter α. Here $\alpha=0$, 0.5, 2.0, 3.0, 5.0, 7.5, and 10.0 (top to bottom). The concentration is $p=0.29$ (i.e., interaction range $R=2$). Here $f=1.234/2$. (b) Same as in (a) but $f=1.260/2$.

TABLE II. Effective $2f$ values for $p=0.30$ ($R=2$).

$\alpha=5.0$		$\alpha=7.5$	
$N=5000$	$N=50\,000$	$N=5000$	$N=50\,000$
1.01	1.21	0.55	1.06

($p_c=0.29$) one can see a straight line with a slope of about 0.62. This is practically identical to the case of $R=1$ (nearest-neighbor hops only). The asymptotic slope, $f=0.616$, is extremely close to that for $R=1$ at a similar number of steps ($N=5000$), where $f=0.618$ (and where with a much larger number of steps, $N=200\,000$, one obtains $f=0.617$). This shows that the two problems ($R=1$ and 2) belong to the same universality class (same f for $S_N \sim N^f$). Furthermore, we see a "crossover to Euclidean behavior" for concentrations above criticality (0.30,0.31,0.32) that is similar to the $R=1$ case. Here, the limiting slopes are all higher than f and actually the lines all "curve up."

To further investigate the scaling exponent we plot[19] S_N^f/N vs N in Fig. 2. In Fig. 2(a) we plotted with $2f=1.234$ while in Fig. 2(b) with $2f=1.260$. For the $\alpha=0$ case (all hops equally probable) we find that the value $2f=1.234$ is slightly better, paralleling our previous investigation for $R=1$. The conclusion drawn is that the *spectral dimension,* which is given by $d_s=2f/(2-d/D)=2f/\frac{86}{91}$, is closer to 1.31 than 1.33, thus giving some preference to the Aharony-Stauffer conjecture, $d_s=2D/(D+1)$, over the Alexander-Orbach-Rammal-Toulouse conjecture, $d_s=\frac{4}{3}$ (in the above case, the Euclidean dimension $d=2$ and the fractal dimension $D=\frac{91}{48}$). For $\alpha>0$, i.e., hopping probabilities that decrease with range, it appears from Fig. 2 that it takes *longer* (more steps) to reach the asymptotic limit. This is expected since it takes, on the average, several nominal time steps before an occurrence of a "long-range" hop.

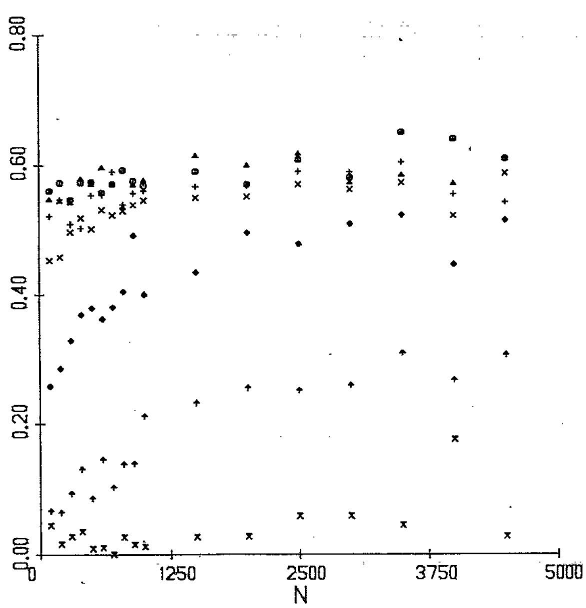

FIG. 3. Plot of $\partial \ln S_N/\partial \ln N$ vs N for different α values. Same values and symbols as in Fig. 2.

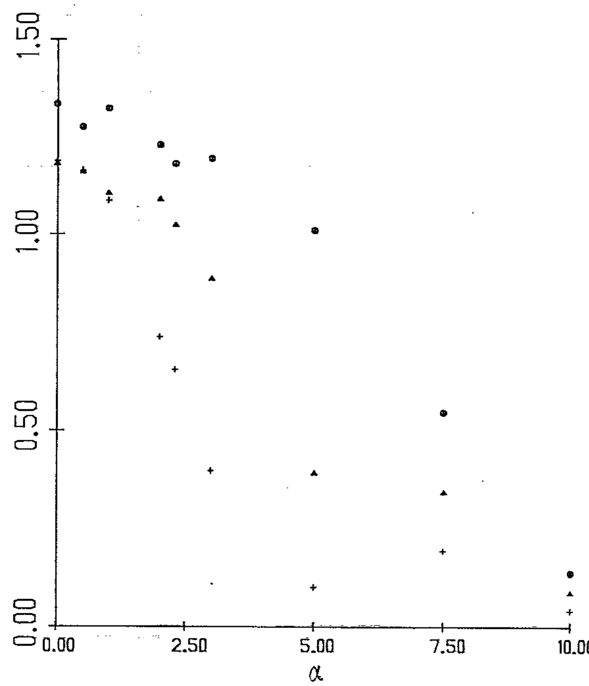

FIG. 4. The short-time random-walk effective fractal exponent $2f$ as a function of α, for three different interaction ranges, $R=2$ (at $p_c=0.30$, circles), $R=3$ (at $p_c=0.16$, triangles), and $R=5$ (at $p_c=0.07$, crosses).

The same is also borne out by Table II, where we compare results for our typical "long" runs ($N=5000$) with those of much longer runs ($N=50\,000$). The larger the value of α (i.e., the less probable the long jumps), the further away from the asymptotic value one finds f (at $N=5000$).

In Fig. 3 we show in more detail the variation of the

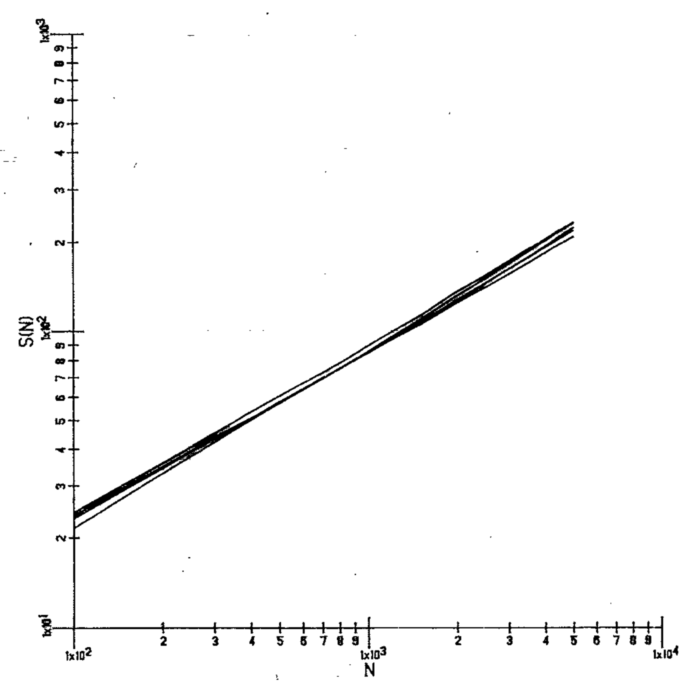

FIG. 5. Plot of S_N vs N in logarithmic form for different interaction cutoff ranges and $\alpha=0$. Here, top to bottom at $N=5000$ steps, $R=5$, $R=1$, $R=2$, $R=3$, and $R=4$, respectively.

"effective" (local) exponents f $(=\partial \ln S_N/\partial n N)$ with time (N), with α as a parameter. We see that for our highest value of α $(=10)$ we get $f=0.05$, compared with 0.62 at $\alpha=0$. In Fig. 4 we show explicitly the dependence of f on α for a number of ranges R. Figure 5, which is for $\alpha=0$, shows a log-log plot of S_N vs N for a number of ranges ($R=1,2,3,4,5$) at *nominal* criticality. The "universality" of the slopes is obvious from the figure. A problem may be the uncertainty in the nominal values of the critical concentrations in Table I. These are less certain for $R>2$.

While Fig. 1 shows crossovers for $\alpha=0$, Table III shows a similar crossover behavior for $\alpha>0$. Here again we see a "time delay" for the crossover ($p>p_c$), similar to the delay in reaching asymptotic f values at $p=p_c$ shown in Fig. 2.

We notice that an r^{-n} dependence for the jump probabilities, with n an integer, is implicitly included in our model if we establish the correspondence with the parameter α. For example, for $R=2$, using $\alpha=2.0$, the exponential form gives probabilities of 0.8815 and 0.1185 for jumps of $r=1$ and 2, respectively. This is equivalent to an r^{-3} form which gives jump probabilities of 0.8889 and 0.1111, for the respective values. Similarly, other correspondences can be established.

In summary, no real surprises can be claimed for the long-range random walk on percolating clusters. Our data indicate that scaling and universality are still intact. However, for a steeply falling-off jump probability some interesting effects are found, indicating an effective percolation threshold and an asymptotic behavior that are approached more gradually in time. We can speculate that for experimental systems, where the cutoff range itself is probably a function of time,[1,3] the actual behavior will also exhibit an effective fractal spectral dimension (i.e., $f<1$). This may already have been observed very recently.[11–13,20]

TABLE III. Effective $2f$ values for $N=5000$ ($R=2$).

p	$2f$ $\alpha=0$	$\alpha=1$	$\alpha=2.3$
0.29	1.23	1.22	1.15
0.30	1.33	1.32	1.18
0.31	1.36	1.35	1.28
0.32	1.46	1.42	1.34

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