Diffusion-driven spreading phenomena: The structure of the hull of the visited territory

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We study the hull of the territory visited by N random walkers after t time steps. The walkers move on two-dimensional substrates, starting all from the same position. For the substrate, we consider (a) a square lattice and (b) a percolation cluster at criticality. On the square lattice, we (c) also allow for birth and death processes, where at every time step, αN walkers die and are removed from the substrate, and simultaneously the same number of walkers is added randomly at the positions of the remaining walkers, such that the total numbers of walkers is constant in time. We perform numerical simulations for the three processes and find that for all of them, the structure of the hull is self-similar and described by a fractal dimension d_H that slowly approaches, with an increasing number of time steps, the value $d_H = 4/3$. For process (c), however, the time to approach the asymptotic value increases drastically with increasing fraction of N/α, and can be observed numerically only for sufficiently small values of N/α.

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I. INTRODUCTION

Diffusion driven spreading phenomena have received considerable interest in the past. The applications range from particle diffusion or diffusion of members of a given species to the spread of diseases and the migration of populations [1–5]. A central quantity of interest is the number of distinct sites (the "territory") $S_N(t)$ visited by N random walkers after t time steps. In particular on two-dimensional substrates, the evolution of S_N with time is quite complex, even in the simple cases when there are no interactions between the diffusing particles and the substrate is homogeneous.

For N random walkers moving on regular or fractal substrates, analytical expressions for the evolution of the territory have been obtained in Refs. [6–9]. When birth and death processes are included [10], an analytical treatment of the territory is still lacking, but the mean distance between the walkers as well as the position of their center of mass could be determined analytically [11].

In this paper, we are interested in the structure of the external perimeter ("hull") of the territory visited by N random walkers moving (i) on a square lattice, (ii) on a percolation cluster at criticality, and (iii) on a square lattice where, at every instant of time, a certain fraction of walkers dies and is removed from the lattice, while simultaneously the same number of walkers is added randomly at the positions of the remaining walkers. For all these cases, the territory $S_N(t)$ cannot be simply reduced to the product $NS_1(t)$ for populations when N is large. Here, $S_1(t)$ is the territory visited by a single random walker [12–16]. For illustrations of the hull for the three cases; see Fig. 1.

It has been argued by Mandelbrot [17] that the hull of the territory of a single random walker moving on a homogeneous $2d$ substrate is self-similar and described by a fractal dimension $d_H = 4/3$, which agrees with the fractal dimension of self-avoiding walks in $d=2$. Here we find that for long enough times, this value also describes the hull for the cases considered here. In (i) and (ii) the convergence of d_H to 4/3 is quite fast, also for large populations, while in (iii) the asymptotic value can only be reached numerically when the fraction between N and α is quite small.

II. MODELS FOR SPREADING PHENOMENA

We have considered three models for the spreading phenomena in $2d$ substrates. In each model, N random walkers start from the same origin. In the first model, the walkers then move independently on a square lattice. In this case, $S_N(t)$ can be calculated analytically for large values of N [7], with the result

$$S_N(t) \sim \begin{cases} t^2, & t \ll t_x \\ t \ln \dfrac{N}{\ln t}, & t_x \ll t \ll t_x' \\ \dfrac{Nt}{\ln t}, & t \gg t_x'. \end{cases} \tag{1}$$

The crossover times t_x and t_x' scale with N as $t_x \sim \ln N$, and $t_x' \sim e^N$. Accordingly, even for small values of N, the "trivial" third regime where the territory increases linearly with N does not occur within a reasonable number of time steps. Hence, apart from the very short initial regime, only regime II is relevant.

In the second model, N random walkers move on a percolating cluster at criticality. Now, above $t_x = \ln N$, $S_N(t)$ scales as [8,9]

$$S_N(t) \sim [\ln(N)]^{(d_s/2)(d_w^l - 1)} t^{d_s/2}, \quad t \gg t_x, \tag{2}$$

where $d_w^l = d_w / d_{min}$ is the anomalous diffusion exponent in topological space, d_w is the fractal dimension of a random walk, d_{min} is the fractal dimension of the shortest path, and

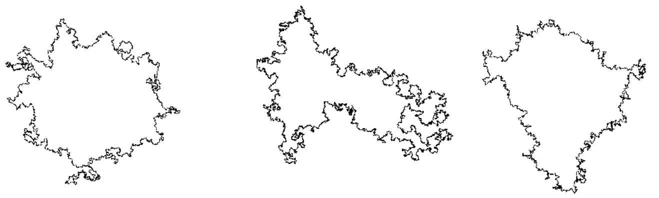

FIG. 1. The forefront of the territory covered by $N=320$ random walkers (a) on a regular square lattice after $t=10^5$ Monte Carlo (MC) steps, (b) on a two dimensional percolating cluster at criticality after 10^7 MC steps, and (c) for the migration model, with α=0.001 and $t=2\times10^5$ MC steps.

d_s is the spectral dimension. For a detailed discussion of these quantities see Ref. [18]. A third regime, where $S_N(t)\sim N$, does not appear here.

In the third model, N random walkers move on a square lattice. At every time step, αN walkers die and are removed from the lattice. Simultaneously the same number of walkers is added randomly at the positions of the remaining walkers [10,11]. The fact that walkers can die everywhere but can be born only at the present locations of already existing walkers, changes the dynamics of the population drastically: The mean square distance between the random walkers increases as [11]

$$R_2^2(t)=\rho_0^2\left[1-\exp\left(-\frac{t}{\tau_0}\right)\right], \tag{3}$$

reaching $\rho_0\equiv\sqrt{4dD\tau_0}$ for $t\gg\tau_0\equiv(N-1)/2\alpha$, where D is the diffusion constant. While R_2 remains constant for $t\gg\tau_0$, the center of mass of the N particles diffuses as [11]

$$R_{cm}^2(t)=\frac{\rho_0^2}{2}\,\frac{t}{\tau_0}. \tag{4}$$

Thus, after an initial spreading period $(t<\tau_0)$, where the center of mass moves as in the case of N independent particles, the particles cluster around their center of mass within a ball of radius ρ_0 and diffuse as a whole. There is no analytic treatment of the territory visited by the random walkers.

III. RESULTS

In the numerical simulations, N particles are initially placed at the center of a large square lattice, on the same site. Multiple occupancy of a site is allowed and there is no interaction between the particles. Periodic boundary conditions are used, although in all calculations the lattice size was large enough in order to exclude finite size effects. After placing the particles at the center of the substrate, we move them by implementing a regular random walk. A particle is chosen at random and chooses with equal probability one of its nearest neighboring sites. If the site belongs to the substrate, the particle moves to that site. Otherwise, it remains at the same position. In any case, time is increased by $1/N$. Accordingly, one Monte Carlo step consists of N attempts to move the particles. We are interested in the way the structure of the forefront of the ‘‘epidemics’’ changes with time. To this end, we continuously monitor the sites which constitute the external perimeter of the visited territory, and we measure its fractal dimension d_H as a function of time, by using the following box counting technique [19,20]. For a given time, the lattice is successively binned in square boxes of different linear size l, ranging from $l=4$ to $l=1000$ and we cover the entire lattice with such boxes. We count the number of boxes $M(l)$ of linear size l which include at least one perimeter site. In practice, a box is considered to include part of the perimeter if it includes parts of both the territory covered and the unvisited area. At a given time t, $M(l)$ scales with l as

$$M(l)\sim l^{d_H}. \tag{5}$$

Accordingly, when $\ln M$ is plotted versus $\ln l$, a straight line is expected with slope d_H.

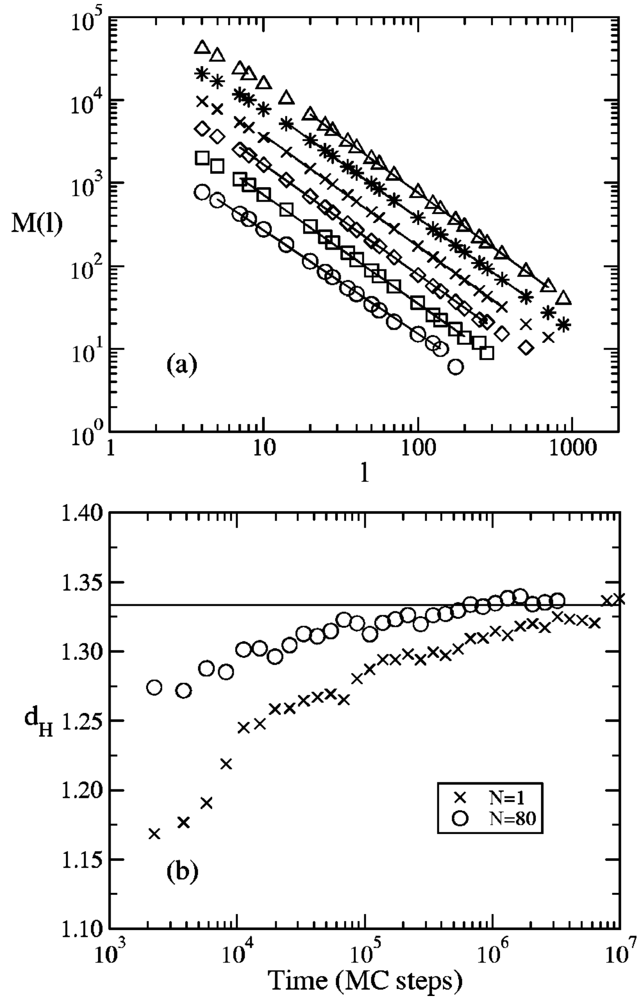

FIG. 2. (a) For the case of N particles performing random walks on a regular square lattice of size 7000×7000, we show $M(l)$, the number of boxes of size l that contain at least a part of the visited area forefront, vs l. Here $N=80$ and results are averaged over 100 realizations. From bottom to top $t=8.2\times10^3$, 3.32×10^4, 1.09×10^5, 3.42×10^5, 10^6, and 2×10^6 Monte Carlo steps. (b) The fractal dimension of the perimeter of the sampled area versus time, for $N=1$ (crosses, 1000 realizations) and $N=80$ (circles, 100 realizations). The asymptotic value is consistent with 4/3.

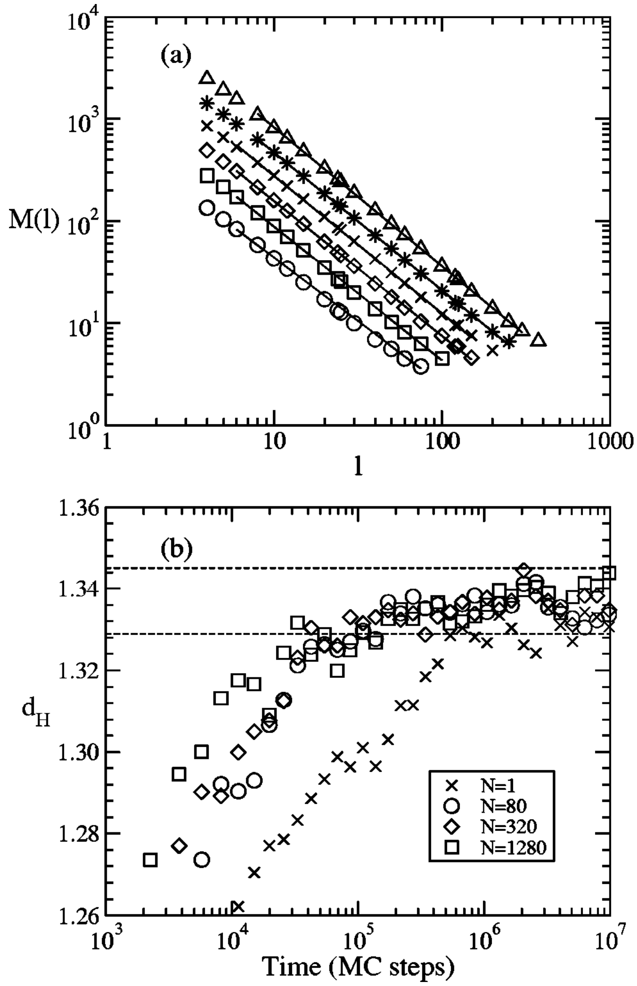

FIG. 3. (a) $M(l)$ vs l for the case of $N=80$ particles moving on a percolation cluster embedded on a 3000×3000 square lattice. From bottom to top: $t=8.2\times10^3$, 3.32×10^4, 1.09×10^5, 3.42×10^5, 10^6, and 3.2×10^6, Monte Carlo steps. (b) The fractal dimension d_H vs time, for $N=1$ (crosses), $N=80$ (circles), $N=320$ (diamonds), and $N=1280$ (squares). Results are averaged over 100 realizations. (For $N=1$, 1500 realizations were used).

A. *N* random walkers on a square lattice

Figure 2(a) shows, in a double logarithmic presentation, the number of boxes of length l, $M(l)$, as a function of l, for t in the range between 10^3 and 10^7 Monte Carlo steps. For large l values all curves are straight lines. The slopes of the lines represent the fractal dimension of the hull, which is then shown in Fig. 2(b). We see that d_H gradually increases with time, from the value 1.25 at $t\simeq10^3$ towards 4/3 at $t\simeq10^5$. This value for d_H is identical to the fractal dimension of a self-avoiding walk (SAW). It has already been conjectured by Mandelbrot [17] that the hull of the territory covered by a single random walk has the structure of the SAW. It is interesting that even though we are well below t'_x, where the territory $S_N(t)$ is composed of the territories $S_1(t)$ of single random walkers, the fractal dimension of the hull is equal to 4/3. Concerning the speed of convergence to $d_H=4/3$, it is obvious from Fig. 2(b) that it is faster for the case of many particles than in the case of a single particle.

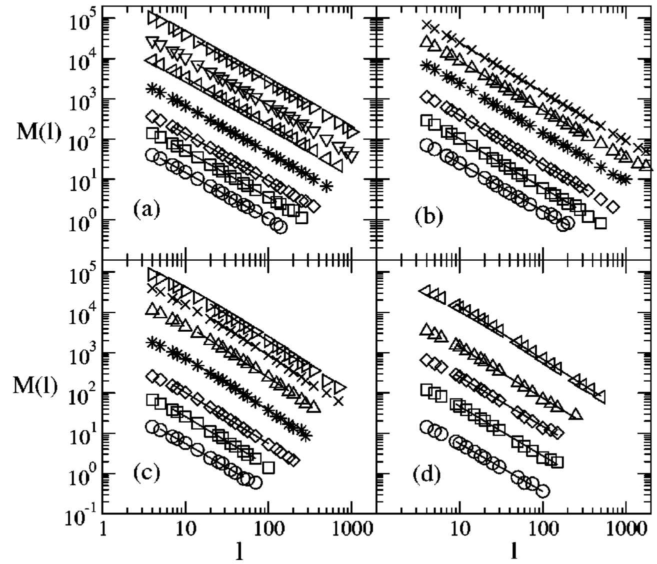

FIG. 4. (a) $M(l)$ vs l for the migration model on a square lattice of size 7000×7000. Here $N=1000$ and $\alpha=0.01$. The symbols correspond to the rescaled simulation data (in arbitrary units), while the continuous lines are the best fits. From bottom to top: $t=8.2\times10^3$, 3.32×10^4, 1.09×10^5, 3.42×10^5, 10^6, 3.2×10^6, and 9.8×10^6 Monte Carlo steps. The slopes of these lines give the fractal dimension d_H of the external perimeter, whose time evolution is depicted in Fig. 5. (b) $N=1000$ and $\alpha=0.1$. (c) $N=10$ and $\alpha=0.5$. (d) $N=8$ and $\alpha=0.5$.

B. *N* random walkers on a percolation cluster

Next we discuss the hull of the epidemics that spreads in the fractal percolating cluster at criticality. Figure 3(a) shows $M(l)$ versus l for several different times. From the slopes of the lines we can deduce d_H. Figure 3(b) shows that already after 3×10^4 time steps the asymptotic value of $d_H=4/3$ is approached, as in the regular lattice case. This result clearly differs from the fractal dimension of a SAW on percolation cluster at criticality, which is $d_F^{SAW}\simeq1.27$ [21,22]. The fractal dimension of the hull of the percolation cluster was found to be $d_H=1.75$ [23,24], but using a slightly different definition

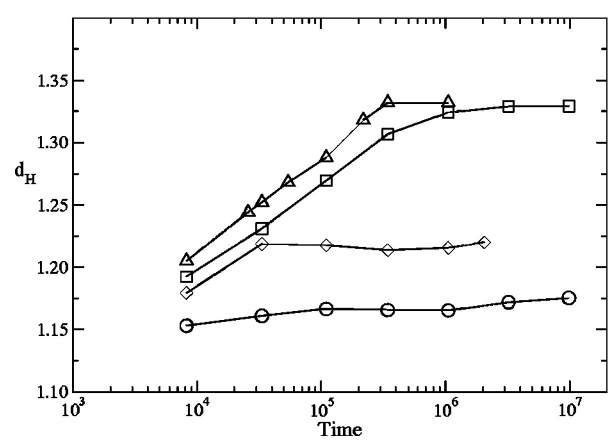

FIG. 5. Time evolution of d_H for the four cases presented in Fig. 4. From bottom to top we have data for $N=1000$ and $\alpha=0.01$, $N=1000$ and $\alpha=0.1$, $N=10$ and $\alpha=0.5$, and $N=8$ and $\alpha=0.5$.

of the hull and regarding next nearest neighbors of the perimeter as connected, Grossmann and Aharony found $d_F^{(Hull)}$ to be $d_F^{(Hull)} \simeq 4/3$ [18,19,25,26]. Thus, the fractal dimension of the hull of the epidemics is the same as the fractal dimension of the external perimeter of the percolation cluster. Furthermore, it turns out from Fig. 3(b) that the bigger the value of N the faster the conversion to the 4/3 value.

C. Migration model

Figure 4 shows, in a double logarithmic presentation, the number of boxes of length l, $M(l)$, as a function of l, for t values between 10^3 and 10^7 time steps. Each panel corresponds to a different value of N/α. From the slopes we obtain the values of the fractal dimension of the hull, which are shown in Fig. 5.

For $\alpha = 0.5$ and $N = 8$ and 10 the fractal dimension of the hull converges, relatively fast, to $d_H = 4/3$. In this case, $\tau_0 = (N-1)/2\alpha$ is quite small, and the asymptotic regime where a "ball" of size ρ_0 diffuses can be seen numerically. Clearly, the structure generated is the territory of a single random walker that moves slowly on a lattice with spacing ρ_0. Hence, for sufficiently large times we expect $d_H = 4/3$. This argument also holds for arbitrary N and α values, but for $\rho_0 \gg 1$, the asymptotic regime cannot be observed within a reasonable number of time steps. In these cases the hull is still a self-similar object, but the fractal dimension can be well below 4/3: $d_H \simeq 1.18$ (for $N = 1000$ and $\alpha = 0.01$) and $d_H \simeq 1.22$ (for $N = 1000$, and $\alpha = 0.1$), for t between 10^5 and 10^7 Monte Carlo steps.

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