Percolation of randomly distributed growing clusters: Finite-size scaling and critical exponents for the square lattice

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(Received 26 April 2010; revised manuscript received 23 June 2010; published 11 October 2010)

We study the percolation properties of the growing clusters model on a 2D square lattice. In this model, a number of seeds placed on random locations on the lattice are allowed to grow with a constant velocity to form clusters. When two or more clusters eventually touch each other they immediately stop their growth. The model exhibits a discontinuous transition for very low values of the seed concentration p and a second, nontrivial continuous phase transition for intermediate p values. Here we study in detail this continuous transition that separates a phase of finite clusters from a phase characterized by the presence of a giant component. Using finite size scaling and large scale Monte Carlo simulations we determine the value of the percolation threshold where the giant component first appears, and the critical exponents that characterize the transition. We find that the transition belongs to a different universality class from the standard percolation transition.

DOI: 10.1103/PhysRevE.82.041108 PACS number(s): 64.60.ah, 61.43.Bn, 05.70.Fh, 81.05.Rm

I. INTRODUCTION

Percolation represents a paradigmatic model of a geometric phase transition [1–7]. It has been widely studied and has numerous applications in many areas of physics [8–17]. Its importance and practical applications are described in detail elsewhere, see for example [2]. Here, we will present for the sake of clarity and completeness, some necessary definitions of important quantities that commonly appear in the literature. In the classical site percolation model, the sites of a square lattice are randomly occupied with particles with probability p, or remain empty with probability $1-p$. Neighboring occupied sites are considered to belong to the same cluster and percolation theory simply deals with the number and properties of these clusters. When the occupation probability p is small, the occupied sites are either isolated or form very small clusters. On the other hand, for large p there are a lot of occupied sites that have formed one large cluster and it is possible to find several paths of occupied sites which a walker can use to move from one side of the lattice to the other. In this latter case, it is said that a giant component of connected sites exists in the lattice. This component does not appear in a gradual "linear" way with increasing p. It appears suddenly at a critical occupation probability p_c. Below p_c there are only small clusters and even if we increase the lattice size considerably, these clusters remain small, i.e., the largest cluster does not depend on the system size. Above p_c, suddenly, small clusters join together to form a large cluster whose size scales with system size. Hence, the term giant component or infinite cluster which is very common in the literature [1].

In percolation, p plays the same role as the temperature in thermal phase transitions, i.e., that of the control parameter, while the order parameter is the probability P_∞ that a site belongs to the infinite cluster. For $p > p_c$, P_∞ increases with p by a power law

$$P_\infty \sim (p - p_c)^\beta. \tag{1}$$

Other important quantities are the correlation length ξ which is defined as the mean distance between two sites on the same finite cluster and the mean number of sites S of a finite cluster. When p approaches p_c, ξ increases as

$$\xi \sim (p - p_c)^{-\nu}. \tag{2}$$

The mean number of sites S of a finite cluster also diverges at p_c

$$S \sim (p - p_c)^{-\gamma}. \tag{3}$$

The critical exponents β, ν and γ describe the critical behavior associated with the percolation transition and are universal. They do not depend on the structure of the lattice (e.g., square or triangular) or on the type of percolation (site, bond or even continuum) [2].

In this paper, we study numerically the percolation properties of the growing clusters model [18] for the case of 2D square lattice. The model is described in Sec. II. We focus on the intermediate concentration regime and find that the model exhibits a nontrivial percolation transition which belongs to a different universality class from standard percolation. We determine quite accurately the position of p_c, and the values of the critical exponents associated with this transition.

II. GROWING CLUSTERS MODEL

The growing clusters model was introduced in detail in [18]. A square lattice of size L is randomly populated with "seeds" with probability p. These seeds are allowed to grow to clusters isotropically and stop when they touch another growing aggregate, see Fig. 1 for a schematic of the system evolution. At every time step, i.e., one Monte Carlo Step (MCS) all seeds are investigated once for the possibility to grow in size instantaneously in all neighboring sites. Investigation sequence is random in order, as is customary in such type of simulations. Sequential investigation leads to statistically the same results. Each seed is allowed to grow its periphery by one layer (increase the radius by one) provided that there is no overlapping with other growing seeds. Thus,

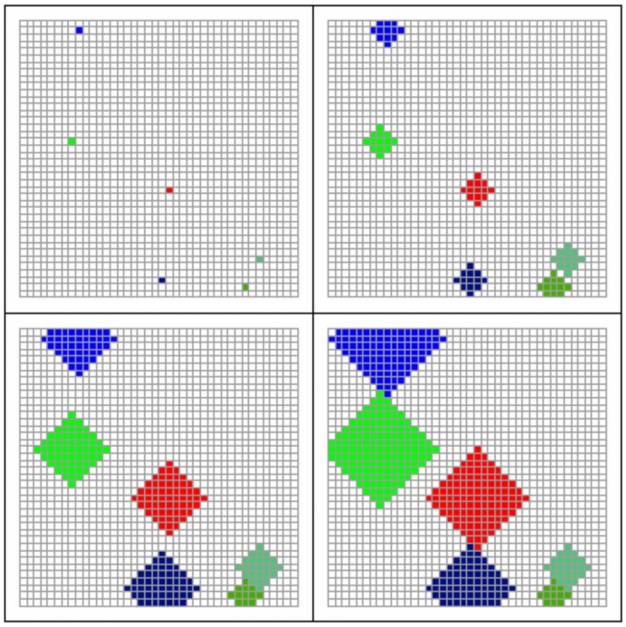

FIG. 1. (Color online) Top Left: at time $t=0$ Monte Carlo Steps (MCS) 6 seeds are randomly placed on a lattice with $L=40$. Top right: after $t=3$ MCS there are 6 evolving clusters. Bottom Left: After $t=6$ MCS there are 4 evolving clusters, as 2 clusters have touched each other and stopped growing. Bottom right: final state of the system. There are no evolving clusters and 3 stable clusters have been formed.

each seed becomes an evolving cluster. As soon as two, or more, clusters touch each other, the growth of all of the adjoined clusters stops. Neighboring sites are considered to belong to the same stable cluster. To study the model we perform large scale Monte Carlo simulations and we monitor the properties of the clusters that are formed. Below we present the details of the Monte Carlo algorithm used for the implementation of the growing clusters model.

(1) For each of the L^2 sites of square lattice of size L a random number $r \in [0,1]$ is drawn from a uniform distribution. A probability p is fixed and if $r <= p$ the site is marked as occupied, otherwise is marked as empty. A unique label is assigned to each occupied site. All unique labels are marked as active.

(2) An occupied site is selected at random. If all of its four neighboring sites are empty, then they become occupied and are assigned the same label as the selected site i.e., they take the same label as their "seed." Otherwise, if one or more of the neighboring sites are occupied no growth process takes place. These occupied neighboring sites are assigned a common label equal to the minimum of their unique labels. This label is marked as inactive.

(3) The growth process continues with the random selection of active labels. At each time microstep an active label is selected. The peripheral sites that belong to the evolving cluster characterized by this label, (i.e., sites that have the selected label and at least one empty nearest neighbor) are give the chance to grow. Growth occurs only if all peripheral sites have nearest neighbors with label equal to that of the site or are empty. In such case all empty neighbors of the peripheral sites become occupied and are given the same label as their "parent."

(4) If at least one of the peripheral sites has a nearest neighbor with a label different from the label of the site then growth process stops. A relabeling is done as follows. We identify the nearest neighbors of all the peripheral sites that have a label different from that of the site. From the set V of these different labels we select the one with the minimum value m. All sites on the lattice that have labels belonging to set V are assigned the new label value m. Then this label value is termed as inactive.

(5) The system stops evolving when there are no more active labels.

We average our results over 1000 different system realizations to obtain better statistics. We apply periodic boundary conditions in order to minimize finite size effects, as is commonly done in large scale Monte Carlo methods.

III. FINITE SIZE SCALING

Equations (1)–(3) are valid for infinite systems close to the critical threshold. In practice, however, it is not possible to use them to calculate the critical exponents with considerable accuracy due to finite size effects. Thus, one has to resort to finite size scaling [2] techniques. Due to the finite size of the lattices that can be simulated the order parameter is expected to depend on the system size. Assuming that we are close to the critical threshold so that the correlation length ξ is comparable to the system size L it can be shown that the probability $P_{\max}$ that a site belongs to the largest cluster follows the scaling relation

$$P_{\max} = L^{-\beta/\nu} F[L^{1/\nu}(p - p_c)], \tag{4}$$

where we have deliberately used the notation $P_{\max}$ instead of P_∞ in order to emphasize the finiteness of the systems. Here F is a suitable scaling function. Similarly, any other quantity varying as $|p-p_c|^x$ is expected to scale similar to Eq. (4) with β replaced by the appropriate exponent x and with a different scaling function F. For example, the mean mass S of all the finite clusters i.e., the mean size of the cluster where a randomly chosen site belongs to, which is an important quantity in classical percolation theory, is expected to follow

$$S = L^{\gamma/\nu} F_1[L^{1/\nu}(p - p_c)], \tag{5}$$

where we denote the scaling function with F_1 to emphasize that it is in principal different from the scaling function in Eq. (4).

At $p=p_c$, these quantities are expected to scale as power law since the scaling function reduces to a simple proportionality constant. This result gives a way to determine the critical exponents. One important characteristic of the standard percolation transition is that exactly at the critical point, the largest cluster has a fractal geometry meaning that the mass of the largest cluster S_1 scales with the system size as $S_1 \sim L^{d_f}$, where the fractal dimension d_f is known to be equal to 91/48. Since $S_1 = P_{\max} L^d$, where d is the dimensionality of the space ($d=2$ in the lattice case) one can easily derive a scaling law relating d_f, β and ν, namely $d_f = d - \beta/\nu$.

IV. RESULTS AND DISCUSSION

To obtain an indication of the system critical behavior, we start by monitoring the size of the largest cluster S_1 as a

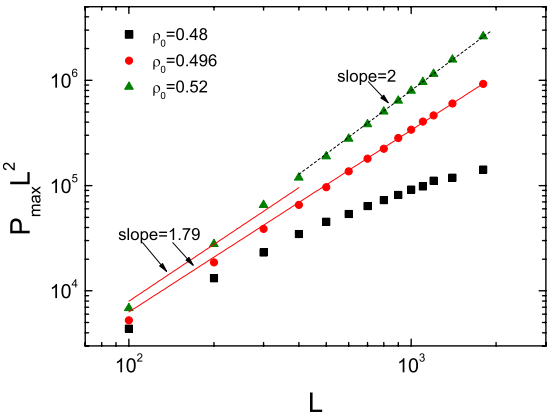

FIG. 2. (Color online) Size of the largest cluster, $S_1 = P_{\max} L^2$, as a function of the lattice size L for three different initial concentrations, below p_c at $p=0.48$ (black squares), at $p_c=0.496$ (red dots), and above p_c at $p=0.52$ (green triangles). Points are Monte Carlo Simulation results. The dotted line has a slope equal to 2 and the red straight line is a power law fit with slope 1.79. The black straight line is just a visual aid.

function of the initial "seed" probability p, see [18] and Fig. 2 therein. There, it is evident that the system exhibits two phase transitions: one very sharp transition at $p=0$ and a second, smoother, transition around $p \simeq 0.5$. The first transition is discussed in detail at [18] and characterized by the fact that the size of the largest cluster S_1, which is the order parameter of the system, has a discontinuity for $p=0$ and, thus, the system exhibits a sharp, albeit artificial, first order phase transition.

The second phase transition turns out to be rather interesting. It is reminiscent of the classical percolation transition and it is important to clarify the extent of this similarity. Thus, we simulate systems of several different sizes and for several different initial seed concentrations. After allowing the growth process to complete and the systems to reach their final states, we study the probability $P_{\max}$ that a randomly chosen site belongs to the largest cluster. When there are only few initial seeds, after the evolution of the system is completed, the clusters formed are small and isolated. However, we expect that with increasing p when a large number of seeds is introduced, the growth process will lead to the formation of a large spanning cluster. This is the classic behavior seen in a system which undergoes a continuous phase transition. In such cases, we can use finite size scaling to determine the position of the critical concentration, p_c, and the critical exponents. Initially, we are interested in the geometry of the largest cluster at criticality and its fractal dimension d_f. In Fig. 2 we plot the size of the largest cluster $S_1 = P_{\max} L^2$ as a function of the lattice size L for three different initial concentrations, namely $p=0.48$, 0.496, and 0.52 (black squares, red dots and green triangles, respectively). For $p=0.48$ (squares) we observe a downward bending curve which clearly indicates that $p=0.48$ is below p_c as S_1 does not scale with the system size for large L. For $p=0.496$ (circles) we observe a straight line implying a power law scaling as expected for $p=p_c$. The value for the critical point agrees reasonably well with our preliminary estimation for the critical point from the position of the maximum of the second largest cluster [18]. We calculate the fractal dimension, d_f, from the slope of the straight line at p_c and find $d_f \simeq 1.79 \pm 0.01$. For $p=0.52$ (triangles) we are well above

FIG. 3. (Color online) Snapshot of the final state of a system with $L=100$ with initial seed concentration $p=0.496$ which is equal to the critical concentration ($p=p_c$). Different colors correspond to different clusters. The final coverage is $p_f=0.538$. The largest cluster is shown in black color.

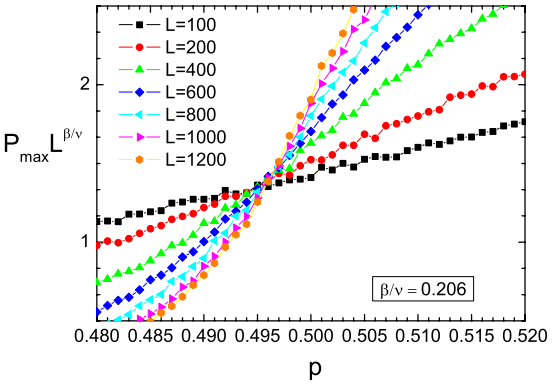

FIG. 4. (Color online) $P_{\max} L^{0.206}$ vs p for seven different system sizes, namely $L=100, 200, 400, 600, 800, 1000, 1200$. The curves cross at $p_c=0.496$ in agreement with the scaling relation Eq. (4).

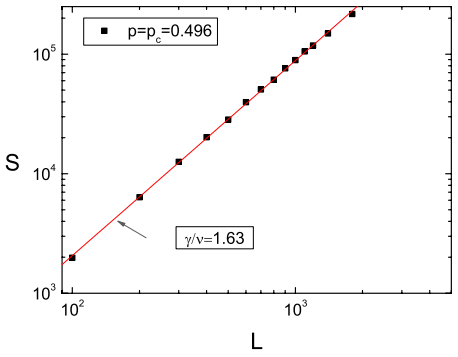

FIG. 5. (Color online) Mean mass S of the finite clusters as a function of L for $p=p_c$. Points are Monte Carlo Simulation results. The straight red line is a best fit to the simulation data and has a slope of 1.63.

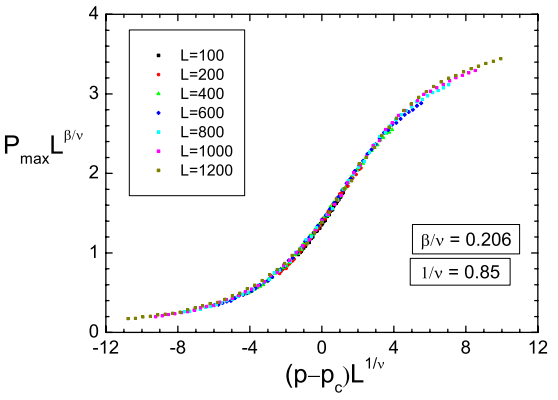

FIG. 6. (Color online) $P_{\max}L^{0.206}$ vs $(p-p_c)L^{0.85}$ for seven different system sizes $L=100,200,400,600,800,1000,1200$. Points are simulation data. The data collapse to one single curve. Thus we estimate $1/\nu=0.85$.

p_c. In the regime above the critical point [1], that the size of the largest cluster exhibits a crossover. For length scales up to the correlation length ξ of the system its structure remains fractal-like and the scaling exponent is still roughly equal to d_f. For length scales greater than ξ the structure appears more compact and the scaling exponent becomes equal to the systems dimensionality, in this case $d=2$. Our findings are consistent with this prediction, as seen in Fig. 2 where the dotted line is a visual aid with slope equal to two. Here, we should point out a technical difficulty. Since the fractal dimension (1.79) is rather close to the system dimensionality ($d=2$), the accurate determination of d_f becomes quite tricky and highly dependent on the accurate determination of the critical point p_c. This is because of the crossover behavior of S_1 for $p>p_c$ which can easily lead to d_f estimations higher that the correct one. Thus, it is necessary to use another method, presented below (see Fig. 4), to confirm that $p_c=0.496\pm0.002$ for the growing clusters model.

Figure 3 shows a snapshot of the clusters that have been formed in a random realization of a growing clusters system with $L=100$ at $p_c=0.496$ once the system has reached its steady state. Periodic boundary conditions have been used for the cluster labeling. The largest cluster is shown in black color.

In order to determine more accurately the percolation threshold and the critical exponent β/ν ratio in the same time, we use Eq. (4). We plot in Fig. 4 $P_{\max}L^{\beta/\nu}$ vs p for seven different system sizes, namely $L=100,200,400,600,800,1000,1200$, and we vary β/ν until all curves cross at one single point only. This is done for $\beta/\nu=0.206(\pm0.003)$ and for $p=p_c\simeq0.496\pm0.002$. This result is in excellent agreement with our previous estimation for the d_f and the scaling relation $d_f=d-\beta/\nu$. It also allows to determine the exact location of the critical point with more accuracy than the previous method.

Figure 5 shows a plot of the mean mass S of all the finite clusters as a function of L for $p=p_c$. At criticality, this quantity is expected to scale as $S\sim L^{\gamma/\nu}$ (see section III). From the slope of the straight line we estimate $\gamma/\nu=1.63\pm0.01$. This, as well as the previous result, are in very good agreement with the scaling relation $d\nu=\gamma+2\beta$.

TABLE I. Comparison of the critical exponents between the classical percolation transition and the transition of the growing clusters model. The exponents τ and σ are associated with the cluster size distribution [1].

Exponent	Class percolation	Growing clusters
β	0.138	0.24
γ	2.38	1.91
ν	1.33	1.17
$\sigma=1/(\beta+\gamma)$	0.395	0.46
$\tau=1+\sigma\nu d$	2.05	2.08
d_f	1.89	1.79

Finally, we calculate the critical exponent ν. In Fig. 6 we plot $P_{\max}L^{0.206}$ vs $(p-p_c)L^{1/\nu}$ for seven different system sizes $L=100,200,400,600,800,1000,1200$, and we vary the exponent $1/\nu$ until all data collapse on one single curve. The data collapse, in agreement with Eq. (4) enables us to determine the critical exponent ν with considerable accuracy. We find that $1/\nu=0.85\pm0.02$.

We can determine other critical exponents from the scaling relations that are known to connect them. Below, for completeness, we present a table with the critical exponents of the growing clusters model in comparison to those of classical percolation (Table I). The difference in the critical exponents shows that the two models belong to different universality classes. Moreover, our calculated value of ν implies that the growing clusters model is also in a different universality class from the "invasion percolation" [11] model.

V. CONCLUSIONS

We have studied the growing clusters model and found that it exhibits two phase transitions with increasing seed concentration. A first order transition at $p=0$ and a continuous transition at $p_c=0.496$, separating a phase of isolated clusters for $p<p_c$ from a phase where a giant component is present for $p>p_c$. Using finite size scaling we have calculated the position of the phase transition and the critical exponents with considerable accuracy to establish that this transition belongs to a different universality class from the standard percolation transition.

ACKNOWLEDGMENTS

This work was supported by the FP6 Project INTERCONY NMP4-CT-2006-033200.

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